On the cellular patterns in thermal convection

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In the theory of thermal convective instability between two horizontal planes there are many solutions that are periodic in the horizontal co-ordinates, while in experiment convection is observed to take place in cellular patterns. It is often assumed, or decided after insufficient argument, that the periodic solutions of the mathematical model 'explain' or correspond to these patterns, but a completely satisfactory discussion of this correspondence has not been given. Indeed, with certain mathematical solutions ambiguities arise as to what cell centres and cellular boundaries are. A detailed discussion has recently become especially necessary because attempts are being made to predict which particular cellular pattern will occur in given experimental conditions.

In this paper the topic is studied afresh and the question is asked: what features, in the mathematical model, correspond to what an experimentalist observes in cellular convective motion? In answer a definition of a cell is formulated which relates certain surfaces in the flow field of the mathematical model to steady vertical cellular boundaries that are observed in experiment, and which shows where the cell centres lie. In particular the classical hexagonal cellular pattern, of the mathematical model, is shown to be the prototype pattern of what is experimentally observed. On the other hand the square and so called 'rectangular' cases of linearized theory are shown not to correspond truly to square and rectangular cells at all. The new formulation is especially relevant to theoretical work on the prediction of cell shape and direction of flow in cells, since precise knowledge of the shapes of the cellular boundaries and locations of cell centres is essential if predictions are to be compared with observation.

1. Introduction

In the theory of thermal convection between two horizontal planes, it is known that there are many possible velocity and temperature fields that are periodic in both of the horizontal co-ordinates. It is generally assumed that such solutions correspond to the convection cells observed by Bénard (1901) and many later experimenters. However, a conclusive discussion of the precise correspondence between the theoretical periodic solutions and the observed cellular patterns does not seem to have been given. Indeed, except for Avsec (1939, p. 148), who pointed out that theoretical ambiguities can arise, most theoretical workers in the field do not seem to have considered that any problem exists. An attempt is made here to expose the problem, and to expound in terms of mathematical models just what it is that experimentalists see. It is emphasized that a discussion of cellular patterns in a mathematical model is *only* relevant in terms of features that are actually observed in convection patterns, and the interpretation of a model must consider those features: the mere fact of periodicity in model and experiment is not sufficient correlation. Moreover, an exposition is essential now that attempts are being made theoretically, by means of non-linear mechanics, to predict the cell shapes that occur in experiment (e.g. those of Tippelskirch 1956).

There are two theories which predict the conditions under which convection takes place. One, initiated by Rayleigh (1916), ascribes the convection to the instability associated with a density gradient in the gravitational field. Instability due to this cause can occur whether the boundary conditions at the confining planes be no-slip (rigid boundary) or zero shear stress (free surface). When the fluid is a liquid and has one free surface, however, Pearson's (1958) theory shows that instability can also occur independently of the gravitational field, because of the variation of surface tension with temperature. There seems little doubt that both theories account for important aspects of the experimental evidence; but from the present point of view we can treat them together, since they both assume solutions of similar form with periodicities in the horizontal co-ordinates. We shall use terminology associated with Rayleigh's theory.

Much of the early theoretical work on the subject was concerned with predicting the critical temperature difference (or Rayleigh number) above which convection may take place, and for a comprehensive account the reader is referred to Chandrasekhar's (1961) book. Such calculations fall within the province of linearized theory, which does not predict the equilibrium velocity amplitude attained by the convective motion or the shape of convection cell most likely to occur. The prediction of such features under given conditions undoubtedly lies within the province of the non-linear mechanics of thermal convection.

It is necessary to recapitulate some of the basic ideas of the non-linear theory, and for this purpose the writer will make reference to a survey article (Stuart 1960); this will be done in the form 'I', followed by page numbers. It has been shown (I, pp. 68, 69) that, for Rayleigh numbers just above the critical value, the convective flow has a spatial form which is dominantly that of linearized theory; the *amplitude* of the convection, however, is not an exponential function of time, as it is in linearized theory, but rather is bounded above and exhibits an equilibrium amplitude towards which the flow tends with the passage of time. These features are indicated by I (equations (2.4), (2.6), (2.7)) when the amplification rate, which is proportional to the difference between the actual and critical Rayleigh numbers, tends to zero. To summarize we may say that the solution of the non-linear problem is almost equivalent spatially to that of linearized theory, but with the vital difference that the amplitude of the non-linear solution becomes steady and finite instead of being exponentially dependent on time. It is this strong spatial connexion between the linear and non-linear theories which gives importance to a detailed study of the notion of cell in the linearized problem.

An additional feature of linearized theory is that many possible spatial patterns ('cell shapes') are equally likely. A discussion of the question of preferment of cell shape, through the action of non-linearity and variation of viscosity with temperature, is given in I (pp. 74–6), by reference to work of Palm (1960), and this is developed further by Segel & Stuart (1962). The results given there are realistic in comparison with observation, namely that in certain circumstances the hexagonal cell is the preferred mode, with fluid at the cell centres flowing in the direction of increasing kinematic viscosity. We emphasize here that in such work it is *essential* to know precisely the locations of cell centres and cellular boundaries of the mathematical model, if meaningful predictions are to be made for comparison with observation.

Let us turn now to the experimental facts. It seems to be agreed among experimental workers that, in most liquids, fluid rises at the centre of each cell and descends at the boundaries, with reverse behaviour in gases. The thing we wish to emphasize here is not the direction of flow, which is to some extent a side issue allowing separate study; but rather that, on the (vertical?) surfaces which are observed and described as cellular boundaries, the vertical velocity apparently has the same sign everywhere. If we are to ascribe observational meaning to a mathematical model, we must introduce this feature into the theory. Another observational feature, that no fluid particle flows through a cellular boundary, has been already incorporated in the theory.

We now refer in detail to some experimental work. In air, smoke is often used to visualize convection cells. In the experiments of de Graaf & van der Held (1953), for example, the authors say that 'In the smoke lying on the bottom circles appeared which. . .became hexagons. The smoke ascended along the boundaries'. It is presumably the presence of smoke in the neighbourhood of the boundaries only which renders the boundaries visible in this early phase. When, later, the smoke has diffused throughout the cell, the boundaries are not rendered so clearly visible by the smoke (Avsec 1939).

In experiments with liquids Silveston (1958, see also Schmidt & Silveston 1959) has used an optical method of visualizing the cell boundaries. Associated with the rising liquid at the cell centres and descending liquid at the cell boundaries is a horizontal temperature gradient; the density is lower at the centres (because of the rising warm liquid) and higher at the boundaries. If a plane vertical beam of incident light is passed through the layer of fluid the light is deflected from the regions of low density towards regions of higher density. Thus the cellular boundaries show up brightly on the screen, while the cell centres are dark (Silveston 1958, figure 13). It is to be emphasized that the method relies on the rising fluid making the centres less dense, and the descending liquid making the boundaries denser. Thus bright annular rings on the screen imply descending liquid everywhere at the associated annular rings in the liquid layer; these regions can thus be described as cellular boundaries. In the actual optical experiments it seems that the incident light was not vertical (Silveston 1958), so the effects are modified. But Silveston's paper implies that the above-described mechanism was important in his experiments.

We see, therefore, that in both liquids and gases experimentalists observe, and describe as cellular boundaries, curves *upon which the vertical velocity has* *one sign only*. In his 1916 paper, Rayleigh recognized this feature, and applied it in an attempt to determine the cellular boundaries for a mathematical model of a square-cell pattern. In so doing, he apparently overlooked an ambiguity, as was pointed out by Avsec (1939), and we shall return to this point later. Our aim throughout this paper is to inject the observational facts, as recorded above, into the theory in order to select from the geometrical flow pattern of a given solution those surfaces, if any, which can represent observed cellular boundaries. The question thus posed is: what features, in the mathematical model, correspond to what an experimentalist observes in cellular convective motion?

2. Theoretical discussion

In order to answer the question posed, we need to discuss the geometrical pattern associated with a given solution of the convection equations. This can be done by considering the equations for the particle paths and filament (streak) lines as discussed for example, by Aris (1962) and by Stuart, Pankhurst & Bryer (1963), namely

$$dx/U = dy/V = dz/W = dt, \tag{2.1}$$

where x and y are horizontal co-ordinates and z is the vertical co-ordinate; U, V, W denote the corresponding components of velocity. The streamlines are defined by (2.1) with $dt \equiv 0$, i.e. with time (t) as a parameter. In steady motion the particle paths and filament lines coincide with the streamlines; but in unsteady motion the streamlines are time-dependent and no longer necessarily coincide with particle paths or with filament lines. This is an important distinction. In general we see that any *steady* solution of (2.1), in the form of a fixed surface, is such that no fluid particles can flow through the surface at any time. Such fixed surfaces, as solutions of (2.1), contain particle paths, filament lines and streamlines and, as we shall see, are of interest to us.

For Rayleigh numbers ($\mathscr{R}$) sufficiently close to the critical value ($\mathscr{R}_c$), (2.1) may have a simplicity that is not present in general. (For a discussion of a related water wave problem see Wehausen & Laitone 1960.) This simplifying feature stems from the nature of the simplest class of solutions in the limiting case of the non-linear problem, when $\mathscr{R}-\mathscr{R}_c$, which is proportional to the amplification rate σ, tends to zero; for a single fundamental mode, W is predominantly of the form

$$W = w(x,y)f(z)A(t), \tag{2.2}$$

where $w(x,y)$ and $A(t)$ are given by the solutions of

$$\left.\begin{aligned} \nabla_1^2 w + a^2 w = 0 \quad &\text{where} \quad \nabla_1^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2, \\ dA/dt = A(\sigma + a_1 A^2 + a_2 A^4 + \ldots) \quad &(\sigma \to 0), \end{aligned}\right\} \tag{2.3}$$

a^2 denoting the sum of the squares of the wave-numbers in the x and y-directions, and a_1, a_2, etc., denoting known constants. In the expression (2.2), W is correct to order $A(t)$, where $A(t)$ has magnitude $\sigma^{\frac{1}{2}}$. Moreover, the $O\{A(t)\}$ parts of U and V are given by

$$U = \frac{1}{a^2}\frac{\partial w}{dx}f'(z)A(t), \quad V = \frac{1}{a^2}\frac{\partial w}{\partial y}f'(z)A(t), \tag{2.4}$$

where the prime denotes differentiation with respect to z. Substituting (2.2) and (2.4) in (2.1), and multiplying by a factor, we have

$$\frac{dx}{(\partial w/\partial x)} = \frac{dy}{(\partial w/\partial y)} = \frac{f'(z)\,dz}{a^2 f(z)\,w} = \frac{f'(z)\,A(t)\,dt}{a^2}. \tag{2.5}$$

The form of (2.5) indicates that the spatial pattern is time-independent and that the streamlines are identical with the particle paths and filament lines; all can be calculated from (2.5). An additional important fact in the present case, however, is that the first (x, y) equation, having coefficients independent of z, can be studied without reference to z. We have in fact

$$\frac{dy}{dx} = \frac{\partial w/\partial y}{\partial w/\partial x}. \tag{2.6}$$

Equation (2.6) has been used earlier by Bisshopp (1960) and Chandrasekhar (1961), but without explanation of its meaning and limitations (see below). From the solution of this equation, $y(x)$ can be substituted into the (x, z) equation of (2.5) and that equation can then be solved for z. The result is

$$f(z) = f(z_0)\exp\left\{a^2\int_{x_0}^{x}\left[\frac{w}{\partial w/\partial x}\right]_{y=y(x)} dx\right\}, \tag{2.7}$$

where x_0, y_0, z_0 specify the streamline. Finally, t can be obtained in the form

$$\int_{t_0}^{t} A(t)\,dt = \int_{x_0}^{x} \frac{a^2}{[f'(z)\partial w/\partial x]_{y=y(x),\,z=z(x)}}\,dx. \tag{2.8}$$

The physical interpretation of this mathematics is that the particle paths, filament lines and streamlines lie on vertical cylindrical surfaces. The curves obtained by intersection of the cylinders with a horizontal plane will, for convenience, be referred to as the 'projected streamlines'; they are defined by (2.6). Formula (2.7) completes the description of the spatial pattern, and (2.8) predicts the time taken for a particle to go between two given positions.

The orthogonal trajectories of the solutions of (2.6) are the curves

$$w = \text{const.} \tag{2.9}$$

Thus, on the projected streamlines, we have as a corollary

$$\partial w/\partial n = 0, \tag{2.10}$$

where $\partial/\partial n$ denotes the derivative normal to the projected streamlines. Our object (for the simple class of flows (2.2)) is to discover which, if any, of these projected streamlines may be regarded as projections of the cellular boundaries. Especially we note that (2.10) must be valid on the cellular boundaries for this simple class of solutions.

In more general cases it may not be permissible to integrate the (x, y) equation of (2.1) separately. Consider, for example, a solution of the non-linear equations consisting of a summation of terms like (2.2)

$$W = \sum_r w_r(x, y) f_r(z)\, A_r(t), \tag{2.11}$$

with corresponding expressions for U and V not necessarily related to (2.4). Moreover, the functions $w_r(x, y)$ and $A_r(t)$ are not necessarily given by (1.3). Since $f_r(z)$ depends, in general, on r, the (x, y)-equation of (2.1) cannot necessarily be integrated independently of z; nor, if A_r depends on r, can they be integrated independently of t. Thus the streamlines, particle paths and filament lines do not necessarily lie on vertical cylindrical surfaces, nor are they necessarily coincident.

There are, however, certain special cases when the simple projected-streamline theory is valid. Then a formula like (2.5) applies and an (x, y)-integration follows straightforwardly. But in such cases $\partial W/\partial n$ is not necessarily equal to zero on the projected streamlines. Moreover, we shall note some cases when *certain* streamlines, particle paths and filaments lie on vertical cylindrical surfaces, even though this is not to be in all parts of the region of flow. In the case considered later these vertical cylindrical surfaces can form the cellular boundaries.

We shall now discuss the projected streamline patterns, governed by equation (2.6) and associated with some of the classical mathematical planforms of the theory of thermal convection. These have been discussed earlier by Bisshopp (1960) and Chandrasekhar (1961), but not, we believe, in a manner that is completely satisfactory from an observational point of view. We again emphasize that the spatial patterns of linearized theory are identical with those of the non-linear theory, in the limit when the Rayleigh number tends to its critical value. In discussing the non-linear problem in this limit, therefore, we can simultaneously give an account of the linearized-theory problem discussed by Bisshopp and Chandrasekhar. Bisshopp's paper is mainly concerned with a classification of cellular solutions of the membrane equation (2.3); we believe, however, that the incidental remarks made there (pp. 379–81) about 'boundaries in an experimental situation,' read in association with the figures 1 *a*, *b*, are unhelpful and perhaps misleading from an observational point of view.

We use co-ordinates and velocity components as defined earlier with lengths referred to h, the distance between the two horizontal bounding planes. As a weak boundary condition on equation (2.3) for w we insist that w shall be periodic in x and y, thus ensuring that w is bounded at infinity. We shall assume that the function $f(z)$ is of one sign, and is the case for the lowest mode instability, and we shall consider convective flows of the simple class (2.2).

Particular solutions of (2.3) that may be found in the literature are

$$\text{(i)} \quad w = \cos kx \cos ky \quad (a^2 = 2k^2), \tag{2.12}$$

$$\text{(ii)} \quad w = \cos kx \cos ly \quad (a^2 = k^2 + l^2), \tag{2.13}$$

$$\text{(iii)} \quad w = 2\cos(lx\surd 3)\cos ly + \cos 2ly \quad (a^2 = 4l^2), \tag{2.14}$$

where k and l are wave-numbers. These three examples are usually described as the square, rectangular and hexagonal cases. From (2.6) the projected streamline patterns corresponding to each of these three cases can be calculated by standard methods (e.g. Stoker 1950); indeed, for cases (i) and (ii), equation (2.6) can be solved exactly (e.g. Bisshopp 1960, equation (28)). The results are indicated in figures 1, 2 and 3, where solid lines and curves indicate (schematically) projected streamlines. The dotted lines and curves represent $w = 0$, i.e. curves of zero vertical velocity. Although, in each case, we have drawn the projected streamlines as stemming from the origin 0, they might equally well have been drawn as entering 0; which case occurs depends on the height z through the

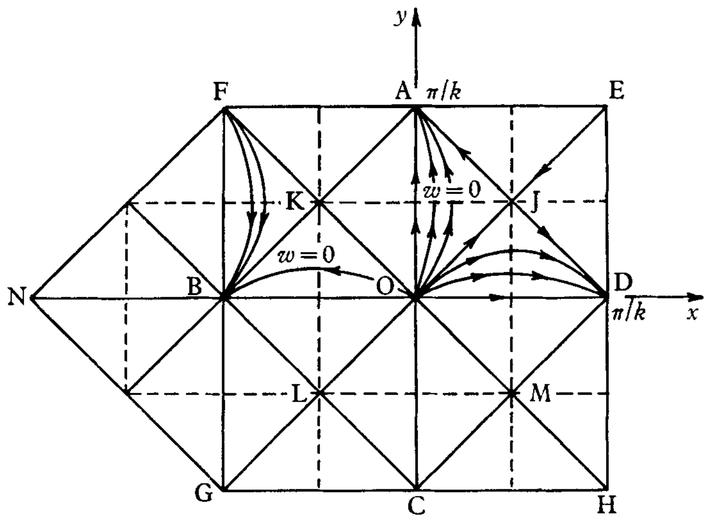

FIGURE 1. Square cell.

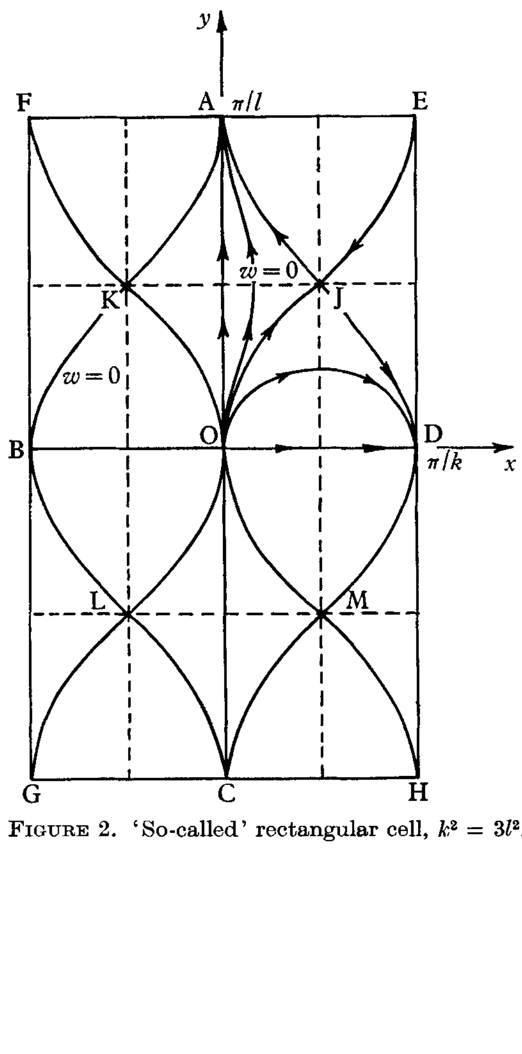

FIGURE 2. 'So-called' rectangular cell, $k^2 = 3l^2$.

amplitude function $f(z)\,A(t)$. We shall return to this point in §5, where we discuss the actual three-dimensional flow in convection cells.

In figures 1 and 2 the points O, A, B, C, D, E, F, G, H and N are nodes of the differential equation (2.6), A, B, C, D being unstable, and the others stable nodes. The points J, K, L, M, on the other hand, are saddle points. In figure 3, O, Q, P are unstable nodes, A, B, C, D, E, F are stable nodes, and G, H, I, J, K, L

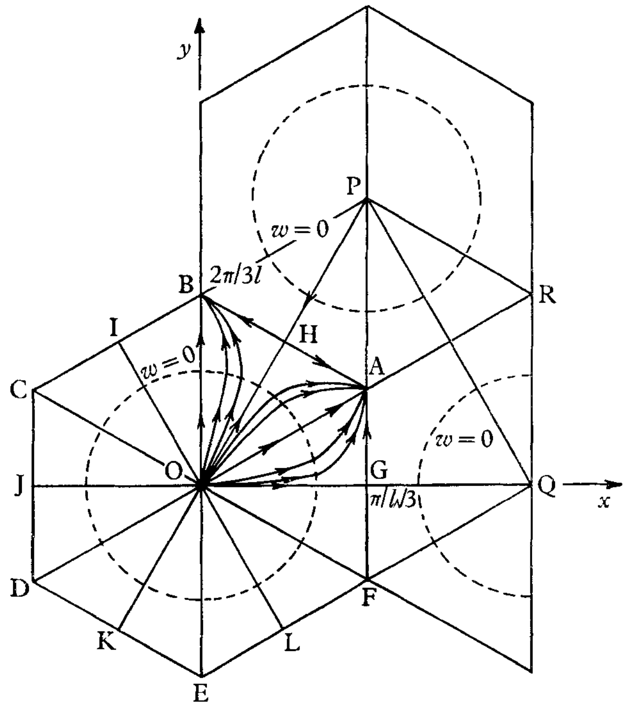

FIGURE 3. Hexagonal cell.

are saddle points. Projected streamlines generally pass from an unstable node to a stable node, but exceptional projected streamlines pass from an unstable node to a saddle point (figure 1, OJ) or from a saddle point to a stable node (figure 1, JD).

Now let us consider the curves $w = 0$. In each of figures 1 and 2 there is a network of dotted straight lines, parallel to the x and y axes, on which $w = 0$. Thus in neighbouring dotted squares (figure 1) or rectangles (figure 2), fluid rises ($W > 0$) and descends ($W < 0$). In figure 3, on the other hand, the situation is quite different. Here we have a set of near-circular closed curves, $w = 0$, none of which intersects another member of the set. In fact, each dotted curve lies entirely within its own bounding hexagon formed from projected streamlines. Thus, in figure 3, near circular regions centred at O, P, Q are 'islands' of, say, rising fluid in a 'sea' of descending fluid. [The sign of W depends on the sign of $f(z)\,A(t)$, $w(x,y)$ being positive at the origin.]

The question we now ask is: what are the features if any, of figures 1, 2, 3, that are consistent with experimental observation? First, an observed cellular boundary has no fluid flowing through it. Now we know that the vertical cylindrical surface, associated with a projected streamline, contains particle paths, filament lines and streamlines, so that no fluid particle passes through such a surface. This suggests that we look for the cellular boundaries among those vertical cylindrical surfaces which have associated projected streamlines. But of this multitude of projected streamlines, which ones are to be considered as likely candidates? To resolve this question, we invoke what seems to be the crucial experimental fact; namely that on the cellular boundary the vertical fluid velocity is observed to have the same sign everywhere.

In the case of figure 3, the answer is quite clear, since there are certain continuous sets of projected streamlines that do not intersect the curves $w = 0$, e.g. the straight lines AB, BC, etc. The hexagon ABCDEF, formed by combination of six of these straight lines, can be a cellular boundary capable of experimental observation. If (cf. (2.2)) $f(z)\,A(t)$ is positive, fluid rises in the dotted near-circle and descends in the region between that closed curve and the hexagon ABCDEF. This could correspond to the cells observed by Silveston in liquids. If $f(z)\,A(t)$ is negative fluid descends in the dotted near-circle and ascends in the region between that closed curve and the hexagon. This could correspond to the cells observed by de Graaf & van der Held in air. We note that, with the present interpretation of the projected streamline pattern, the hexagon BOFQRP is not acceptable as a cellular boundary, because it intersects curves $w = 0$. Moreover, we reject Chandrasekhar's remark (1961, p. 50) that 'The pattern which we considered as hexagonal can be considered, equally, as triangular [since] the unit cell is then represented by the equilateral triangle [OQP]'. Such a triangle cuts the curves $w = 0$, and is not, therefore, what would be observed and described as a cell. Our present point of view is that the form (2.14) for w is associated with a *unique* set of cells, which would be observed by the methods described earlier; the cells are the hexagons with centres at O, P, Q, etc., other hexagons, triangles or shapes being unacceptable.

Let us turn now to figures 1 and 2. In these figures there are no projected streamlines that do not intersect the dotted lines on which $w = 0$. Consequently we can assert that the flows represented by (2.12) and (2.13) do not correspond to patterns that have been observed and described as cellular. Let us now consider Rayleigh's interpretation. In the case of (2.12) and figure 1, he suggested that square ABCD should be regarded as a cellular boundary. It is instructive to note the feature of the boundary ABCD that led Rayleigh to choose it rather than one of the others, as indicated by the following quotation (Rayleigh 1916, p. 443): 'An experimental determination of [a^2 of our first equation (2.3)] might be made by observing the time of vibration under gravity of water contained in a trough with vertical sides and of corresponding section, which depends upon the same differential equations and boundary conditions. The particular vibration in question is not the slowest possible, but that where there is a simultaneous rise at the centre and fall at the walls all round, with but one curve of zero elevation between.' The equation to which Rayleigh refers is our first equation (2.3) and the boundary condition is (2.10). Rayleigh thus selected a boundary for which the vertical velocity is everywhere of one sign (though in practice he permitted zeros). This condition permits the selection of ABCD, as a boundary of a cell centred at O, but it does not exclude other possibilities. For example, the square OFNG is equally possible as a boundary, but of a cell centred at B. We are thus led to an ambiguity: the pattern of figure 1 can be regarded either as a set of convection cells centred at O, E, F, G, H, etc., or as a set of cells centred at A, B, C, D etc., As is indicated by the quotation above, Rayleigh's criterion to distinguish a cellular boundary is clearly the same as the one favoured here (provided we ignore his allowing of zero of w). But we believe, contrary to Rayleigh's remarks, that this criterion, when applied to (2.12) and figure 1, shows that there are no cellular boundaries that would correspond to observation. The resolution of the ambiguity of two possible sets of cells seems to be, therefore, that (2.12) is not a valid mathematical model of what would be observed and described as a pattern of square cells; in our view there are no cells in this observational sense.

In contrast to the discussion of Rayleigh, those of Bisshopp and Chandrasekhar are quite misleading from an observational point of view, since both have a figure that is apparently equivalent to the square AODE of our figure 1, Bisshopp (1960, figure 1*a*) describing it as 'The square cell' and Chandrasekhar (1961, figure 6) as 'The streamlines in the horizontal plane for the square cell'. In no sense does AODE seem to correspond to what experimentalists observe and describe as a cell.

Let us turn now to (2.13) and figure 2. In the literature this is often described as a 'rectangular cell'. For example, Chandrasekhar (1961, p. 45), with $k = 2\pi/L_x$ and $l = 2\pi/L_y$, describes the cells associated with (2.13) as 'rectangles with sides of lengths L_x and L_y'. In our terminology these lengths are $2\pi/k$ and $2\pi/l$, so that Chandrasekhar appears to suggest that the rectangle EFGH (figure 2) should be regarded as a cell; on the other hand Chandrasekhar's figure 5 (1961, p. 46) is apparently a sketch corresponding to the rectangle AODE of our figure 2, and is labelled 'The streamlines in the horizontal plane for a rectangular cell' (see also Bisshopp 1960, p. 380, figure 5, where a similar diagram is labelled 'The rectangular cell'). Both of these rectangular boundaries suggested by Chandrasekhar cut the dotted lines $w = 0$, as can be seen from figure 2. Consequently we assert that (2.13) is not a valid mathematical model of a cellular pattern that could be observed and described as rectangular.

We see that, in figure 2, there are no curves that do not intersect the dotted curves $w = 0$. But there are curves on which w has one sign, if zeros are permitted; curves such as DJA for example, are of this type, and correspond to straight lines such as DJA in figure 1. It might be suggested, therefore, that we could select the closed curve AKBLCMDJ as a cellular boundary, with O as the cell centre. But equally we could select the curve EJOMH, with its mirror image in the line EDH. We are thus led to an ambiguity similar to that of figure 1, namely, that figure 2 may be regarded as representing *either* a set of cells centred at O, E, F, G, H, etc., *or* a set of cells centred at A, B, C, D, etc. The resolution of the ambiguity seems to be that there are no projected streamlines in figure 2 that correspond to what in experiment have been observed and described as cellular boundaries; therefore, in our view, there are no cells in this observational sense.

Having discussed figures 1, 2 and 3 in some detail, we are led inevitably to the point of view that, while figure 3 may be regarded as a mathematical prototype of a convective pattern of cells of the type that experimentalists observe and describe as such, figures 1 and 2 may not be so regarded. The essential, prototype feature of figure 3 is that there are separate closed curves $w = 0$, between which certain projected streamlines pass. The latter projected streamlines, e.g. those forming the hexagon ABCDEF, may be regarded as the projections of cellular boundaries. In figures 1 and 2 there are no separated closed curves $w = 0$, there are no projected streamlines that do not intersect curves $w = 0$, and we are left with the ambiguity described earlier. As a corollary we assert that (2.12) and (2.13) are not mathematical models of what would, in observation, be described as square and rectangular cells.

There are at least two questions that derive immediately from this discussion. First, can the forms (2.12) and (2.13) be altered slightly so as to bring figures 1 and 2 into the prototype form exemplified by figure 3? The answer is that in certain circumstances, at any rate, they can, as shown in §4. Secondly, if we introduce higher-order amplitude terms into (2.14), so that we relax the limit $\sigma \to 0$, can the vertical cylindrical surface of projection ABCDEF still represent a cellular boundary? The answer is that we have shown this to be the case if second-order terms in amplitude are included, as described in the next paragraph.

Typical non-linear calculations of a convective flow field, but valid only to second order in amplitude, have been done with the artificial free-free boundary conditions of zero shear stress at upper and lower boundaries. According to Palm (1960), with his corrections and additions as quoted by Segel & Stuart (1962), the vertical velocity W for a hexagonal cell is of the form

$$W = [A(t)\sin\lambda z + B(t)\sin 2\lambda z]\,[2\cos kx\cos ly + \cos 2ly] + C(t)\,[2\cos kx\cos 3ly + \cos 2kx]\sin 2\lambda z, \quad (2.15)$$

where A, B, C represent Palm's A_{111}, A_{022}, A_{202}, and $k = l\sqrt{3}$. Whereas A is of first order in amplitude, B and C are of second order and are given by

$$B = (-M)^{-1}[H\gamma A + GA^2],$$
$$C = -4P_1A^2, \quad (2.16)$$

where M, H, G and P are constants and γ is a small parameter representing the variation of viscosity with temperature. The variation of A with t is described in Segel & Stuart (1962) as the function $Z(t)$. Using (6.10) and (6.11) of Palm (1960), since they are valid for our purpose, we find the following for the horizontal velocity components U and V:

$$U = -(k\lambda/2l^2)\,(A\cos\lambda z + 2B\cos 2\lambda z)\sin kx\cos ly - (\lambda C/k)\,(\sin kx\cos 3ly + \sin 2kx)\cos 2\lambda z, \quad (2.17)$$

$$V = -(\lambda/2l)\,(A\cos\lambda z + 2B\cos 2\lambda z)\,(\cos kx\sin ly + \sin 2ly) - (\lambda C/l)\cos kx\sin 3ly\cos 2\lambda z. \quad (2.18)$$

Substitution of (2.15) to (2.18) in (2.1) indicates that the projected-streamline equation (2.5) is not generally valid. But it is possible to show that some steady vertical cylindrical surfaces, in particular those of which hexagons such as ABCDEF of figure 3 are projections, do satisfy equations (2.1) for all z and t. Consequently, these surfaces contain particle paths, filament lines and streamlines, even though not all solutions are of this type. Moreover, it can be shown that $W \neq 0$ on surfaces such as that whose projection is ABCDEF: although the surfaces $W = 0$ are no longer vertical cylinders, they do not intersect those vertical cylindrical surfaces which, for $\sigma \to 0$, we found to be the cellular boundaries. Consequently the latter may still be regarded as cellular boundaries. We may say, therefore, that to second order in amplitude, the prototype picture of figure 3 is essentially unchanged, at least for free-free conditions. (In the cases of figures 1 and 2, the inclusion of higher-order amplitude terms would not necessarily remove the ambiguity inherent there.)

We come next to the definition of a cellular boundary. In the past it has often been defined to be (among other things) a vertical surface of symmetry on which the normal gradient of vertical velocity vanishes (cf. Chandrasekhar 1961, p. 43). After our discussion in terms of particle paths, such conditions of symmetry do not seem to be primary ones; rather the condition that no particle passes through the cellular boundary seems more natural. (The earlier conditions probably arose from a more restrictive discussion in terms of linearized theory, since those conditions derive from ours in that case.) In addition we wish to incorporate the observational fact noted earlier that the vertical velocity on a (vertical) cellular boundary is everywhere of one sign. In the next section we formulate a definition of what, in a mathematical description, corresponds to a surface which experimentalists observe and describe as a cellular boundary. Although we restrict attention to what have often been observed, namely cellular boundaries that are steady vertical cylinders, it is not clear that they *must* be of this form. It may be noted here that Veronis (1959) has given a discussion of the flow patterns in a more complex case, that of convection cells in a rotating system.

3. A definition

If the mathematical model of the flow field has steady vertical cylindrical surfaces satisfying the conditions given below, then those surfaces correspond to cellular boundaries that might be observed and described as such.

(i) No particle of fluid passes through a cellular boundary surface at any time.

(ii) The vertical velocity on a cellular boundary surface must be everywhere of one sign and must not be zero at any point except on the two horizontal, or nearly horizontal, bounding surfaces.

As a corollary of (i) we note that the boundary surface must contain particle paths, filament lines and streamlines.

For a single mode of linearized theory, or of non-linear theory to order $A(t)$, i.e. with $\sigma \to 0$, the result $(\partial W/\partial n)_s = 0$ applies, where $(\partial W/\partial n)_s$ denotes the derivative of the vertical velocity normal to the boundary surface. In that simple case, (i) and symmetry imply this result. It seems unnecessary, however, to *impose* $(\partial W/\partial n)_s = 0$ as an independent condition in more general cases;

but we note that for the hexagonal cell, with free-free conditions and to second order in amplitude, the statement is valid on the cellular boundaries. It is likely that, in some cases, conditions (i) and (ii) may define cellular boundaries which are realizable in experiment and on which $(\partial W/\partial n)_s \neq 0$.

It is noted that, because of (ii), we exclude consideration of the physically unimportant higher modes, theoretically possible at much higher Rayleigh numbers, because $f(z)$ changes sign for such modes. We restrict attention to the lowest mode, for which $f(z)$ is of one sign, because this is the case observed experimentally.

4. Further theoretical discussion

For the mathematical models with velocity fields (2.12) and (2.13), it has been demonstrated that there are no surfaces that would, in experimental observation, be described as cellular boundaries. In actual experiment, flow fields rarely have

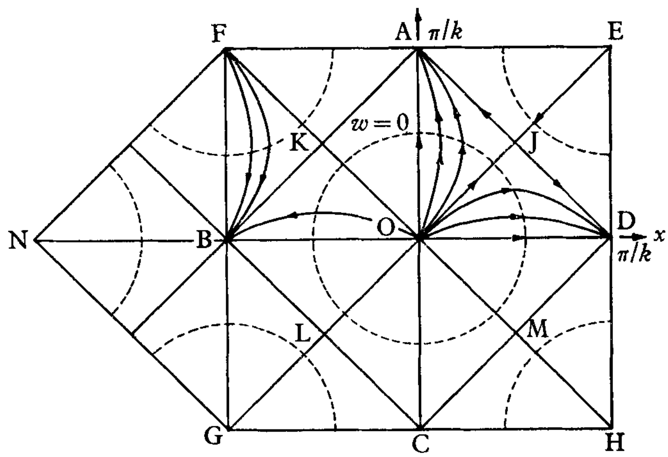

FIGURE 4. Modified square cell.

the relatively simple character envisaged in mathematical theory, and it is natural to suppose that more complicated models might yield cellular boundaries in cases corresponding to (2.12) and (2.13), that are satisfactory from the observational point of view. One would then expect figure 1 to be converted into a form such as is shown in figure 4; a diagram of this kind was first given by Avsec (1939, p. 148). The present writer has not constructed a mathematical model for figure 4 that is completely satisfactory, but he has done so for a case which is a generalization of the so-called 'rectangular' case (2.13), when $k = l\sqrt{3}$. We therefore concentrate on that example.

Segel & Stuart (1962) discuss a solution of (2.3) for $w(x, y)$ of the form (case V of that paper)

$$w = \cos kx \cos ly + \tfrac{1}{2}D \cos 2ly, \tag{4.1}$$

where $k^2 = 3l^2$ and the first component of (4.1) has been normalized to unit amplitude. The overall wave number of (4.1) is $2l$ and $D = 1$ represents the hexagonal cell discussed in §2. We shall consider only D positive, since negative values can be treated by translation of axes. The form (4.1) occurs naturally in the non-linear theory to represent a steady velocity field, when the viscosity is allowed to vary with temperature. This is an example where the incorporation of a small effect ($D \neq 0$) can bring about significant qualitative changes.

When $D < \frac{1}{2}$, it can be shown that the streamline pattern is essentially similar to that of figure 2. But the curves $w = 0$, however, show a significant difference. Instead of the rectangular mesh (dotted) of figure 2, we have curves lying entirely

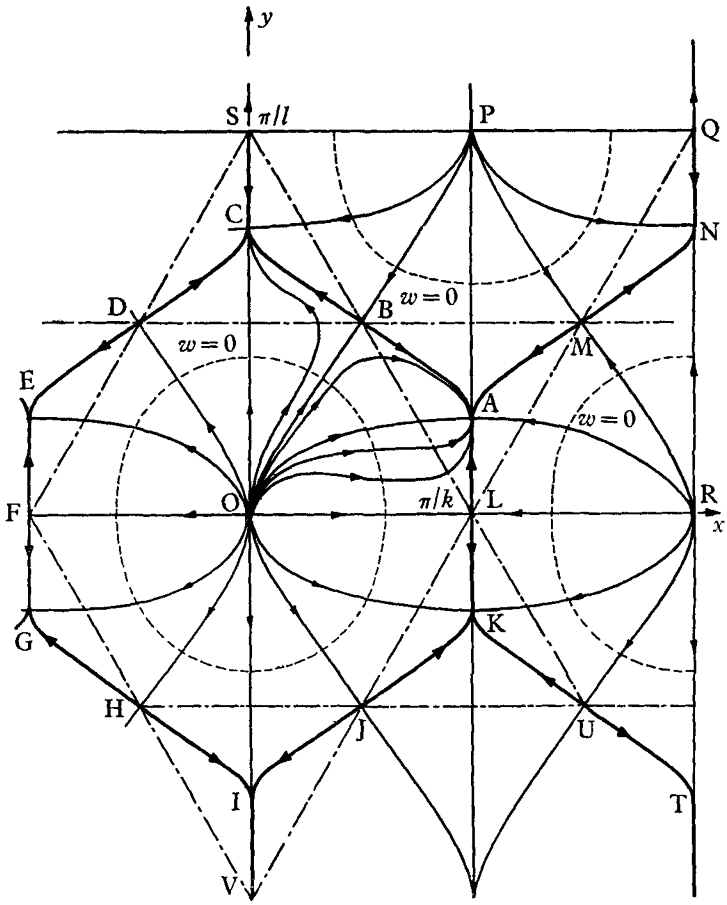

FIGURE 5. Modified rectangular cell. Case of $D = 1/\sqrt{2}$.

within certain projected streamline contours, for example within DJAKBLCM. We may regard this contour as the boundary of a cell centred at O. Similarly E, F, G and H may be regarded as cell centres, but the points A, B, C and D may not be so regarded because *their* corresponding boundaries cut the curves $w = 0$. Thus the ambiguity of figure 2 has been removed, even for very small D, and the resulting pattern corresponds to a unique set of cells.

When $\frac{1}{2} < D < 1$, a different pattern of projected streamlines occurs, and a typical case ($D = 1/\sqrt{2}$) is shown schematically in figure 5. It can be shown that w is everywhere negative within chain-dotted triangles such as BDS and therefore on projected streamline contours such as ACEGIK, and such contours may be regarded as cell boundaries. The curves of $w = 0$ are shown dotted. One slight ambiguity remains in the form of figure 5; we have assumed in the arguments leading to this figure that the streamline BC is parallel to the y axis at C, whereas it might have zero gradient. If the latter were the case the diagram would be modified in the neighbourhood of C and like points.

For $D > 1$ the cellular boundaries have somewhat different shapes, but we shall not pursue these cases further, mainly because Segel & Stuart's (1962) paper indicates that they are less important.

It is shown in Segel & Stuart's paper that the cellular pattern (4.1) is 'stable', in the sense defined there, if $|D| \leqslant 1$. Now, since we know that the origin, O, is definitely a cell centre (while B and D of figure 2 cannot be in the present formulation), we can calculate the direction of flow at the cell centres for the dominant (i.e. first-order) part of the non-linear solution. We find from Segel & Stuart (1962) that the flow is upwards at the centre if the kinematic viscosity decreases as the temperature increases (as in most liquids), and downwards at the cell centres if the kinematic viscosity increases with temperature (as in most gases). These directions of flow are the same as those obtained by Segel & Stuart (1962) for the hexagonal cell (with centre at O in figure 3), and are in accordance with experiment. Moreover, since figure 5 contains cells of roughly hexagonal shape (and this would be true for $\frac{1}{2} < D < 1$), these solutions as well as the standard 'hexagonal' solution might correspond to the hexagonal cells observed in experiment. Thus, in terms of the parameter q of Segel & Stuart's paper, the range of solutions corresponding to observed stable hexagonal cellular patterns has been extended to lower values of q and therefore to larger values of the Rayleigh number. (The parameter $q = a\epsilon^{-\frac{1}{2}}$, where a is proportional to the gradient of kinematic viscosity with temperature and ϵ is almost proportional to the difference between actual and critical Rayleigh numbers.)

The reasoning given in the preceding paragraph is explained by Segel & Stuart (1962), who discussed the hexagonal case without a complete discussion of why O of figure 3 can be a cell centre, while A cannot be. The definition given in this paper clarifies their work, by showing precisely the locations of cell centres and cellular boundaries in the mathematical model, and thereby permitting comparison with observation of the predicted flow directions and cell shapes.

5. The flow in convection cells

In the absence of some specific interpretation, figures 1–5 may be misleading as to the actual flow patterns. If we regard the projected streamlines as real streamlines in two dimensions, we are forced to inquire as to the fate of fluid arriving at the stable nodes. In fact, of course, no such question arises, because the projected streamlines must be regarded merely as a guide in planform to the *three-dimensional* flow patterns.

An attempt is made in figure 6 to illustrate actual streamlines, particle paths and filament lines (which are all coincident) for the (steady) hexagonal cell of figure 3. The curve DOA represents two particular curved projected streamlines OA and OD, from the unstable node O to the stable nodes A and D. The cylindrical surface, of which DOA is the projection, is shown in figure 6 as DOAA′O′D, and the curves shown schematically are streamlines lying in that surface. (The dotted straight lines represent the boundaries, in perspective, of the hexagonal convection cell.) The streamlines are closed curves and only the boundary streamlines, such as OAA′O′, pass through the nodes of the projected picture of figure 3. Streamlines can be drawn in any cylindrical surface corresponding to a given

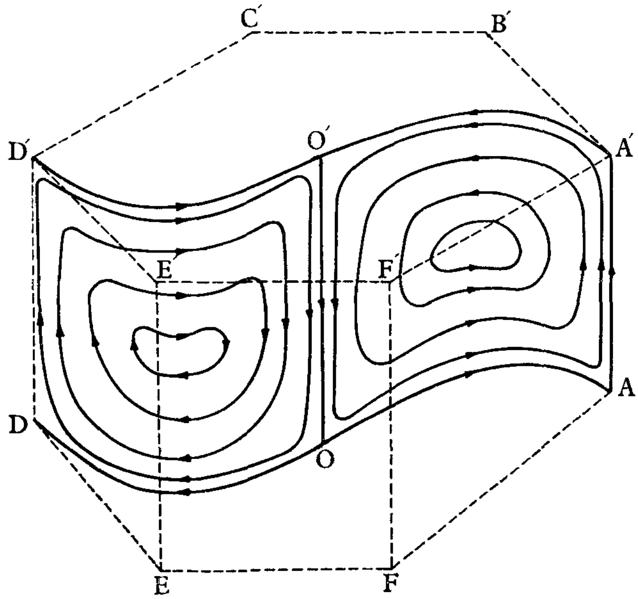

FIGURE 6. Streamlines in a hexagonal cell.

projected streamline, those in figure 6 being typical. For the straight projected streamline, OG, OB of figure 3, the corresponding vertical cylindrical surfaces are plane; streamlines have been computed by Reid & Harris (1959) and are illustrated in their paper and in Chandrasekhar's book (1961, p. 51). For curved cylindrical surfaces computations may be based on equations (2.6) and (2.7) of this paper, $f(z)$ and $w(x, y)$ being given: the time taken for a particle to pass between two given points is then determined by (2.8). Whether O is taken to be an unstable or stable node of the projected streamline pattern is now seen to be immaterial, since it merely reflects the particular height z of the plane considered.

Although the direction of flow in figure 6 is downwards at the cell centre (as in most gases in Rayleigh thermal convection), the streamlines would be similar if the flow were upwards at the cell centre (as in most liquids). The sign of the vertical velocity is the same at all centres in figures 3 and 5 and theories such as those of Palm (1960) and Segel & Stuart (1962) yield correct results for the sign (for figures 3 and 5).

It seems likely that there are cases in which the cellular boundaries and the surfaces $w = 0$ are not steady vertical cylinders. Such possibilities remain to be investigated. It remains also to evaluate fully the effects of finite amplitude beyond velocity corrections of second order. It is emphasized that here we have considered only the simplest convection patterns. Veronis's paper indicates some of the complexities which arise when, for example, rotation is included.

6. Conclusions

(i) A discussion of the experimental observations has led to a definition (§3) of those surfaces which, in a mathematical model, correspond to what are observed and described as cellular boundaries in thermal convection. Attention has been confined to the case where the latter are steady and vertical.

(ii) The classical hexagonal cell of linearized theory has been shown to be the prototype mathematical pattern, irrespective of its special shape, of cellular patterns that are observed experimentally. The quintessence of the prototype pattern is that the vertical cellular boundary, through which no particle of fluid passes, encloses, but is not intersected by, a surface on which the vertical velocity is zero.

(iii) If second-order amplitude effects are included, the same vertical hexagonal cylinders as in linearized theory form the cellular boundaries in the hexagonal case, at least for free-free boundary conditions. It may be conjectured that this result will be true with much larger finite-amplitude effects, present at larger Rayleigh numbers, and for more realistic boundary conditions.

(iv) Attempts at interpretation of the classical square and so-called 'rectangular' cases of linearized theory have been shown to lead to ambiguities; these cases are not of the prototype form, and are not valid mathematical models of square and rectangular patterns. We assert that, for these models, there are no cells in the present observational sense.

(v) In a special case, the mathematical solution of so-called 'rectangular' kind has been modified by a perturbation that can be as small as we please, to yield a set of cells of curved, not rectangular form. The perturbation is associated with the small, but non-zero, variation of viscosity with temperature.

(vi) The definition has been shown to be vital to give an unambiguous statement of cellular boundaries and cell centres in hexagonal and other cases, *so that comparison with observation can be made as to the predicted flow directions and cell shapes.*

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